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COHOMOLOGY OF OPERATOR ALGEBRAS, III. REDUCTION TO NORMAL COHOMOLOGY

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1. Introduction

We continue our study, begun in [7], [8], of the (topological) cohomology of operator algebras. We consider two cohomology theories, the norm continuous, and the normal (ultraweakly continuous), the corresponding cohomology groups being denoted by H_c^n and H_w^n respectively. In our terminology and notation, we follow [7]. The two earlier articles in this series are concerned primarily with the norm continuous case. The present paper deals mainly with normal cohomology, and with its relationship to norm continuous cohomology. We prove, in Sections 5 and 6, that $H_w^n(\mathfrak{A}, \mathcal{M}) = H_c^n(\mathfrak{A}, \mathcal{M}) = H_c^n(\mathfrak{A}^-, \mathcal{M})$, whenever $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space and $\mathcal{M}$ is a dual normal $\mathfrak{A}^-$-module. As an application of these results, we show (Corollaries 6.5 and 6.4) that $H_c^n(\mathcal{R}, \mathcal{M}) = 0$ whenever the von Neumann algebra $\mathcal{R}$ is either type I or hyperfinite, and $\mathcal{M}$ is a dual normal $\mathcal{R}$-module; this had been proved previously in the particular case in which $\mathcal{M} = \mathcal{R}$ ([7], Theorem 4.4; [8], Theorem 3.1; *see* also [5], Proposition 7.14).

In developing normal cohomology theory and relating it to the norm continuous case, we need an extension theorem for n-linear mappings, from a product of (concretely represented) C^*-algebras into a dual Banach space, which are (separately) continuous relative to the ultraweak and weak * topologies. Specifically, we prove, in Section 2, that each such mapping extends, retaining the same continuity, to the product of the corresponding von Neumann algebras. Although an extension process of this type was used during the proof of [8] (Theorem 2.1), it was possible in that particular situation to avoid the need of the full form of the extension theorem.

At one point we make use of a result of TAKESAKI [9] (Corollary 1), which gives a characterisation of the ultraweakly continuous linear functionals on a von Neumann algebra. Although originally stated in a slightly different form, Takesaki's theorem is (easily seen to be) equivalent to the following assertion : a bounded linear functional on a von Neumann algebra $\mathcal{R}$ is ultraweakly continuous if, and only if, it is completely additive on projections. The proof given in [9] exploits the properties of the universal representation of $\mathcal{R}$. In Section 3, we give a proof within the framework of von Neumann algebra theory. For completeness, we include an account of (the essentials of) the original argument used in [9].

With $\mathfrak{A}$ a C^*-algebra and $\mathcal{M}$ a two-sided dual $\mathfrak{A}$-module, it was proved, in [7] (Theorem 3.4), that each ρ in $Z_c^n(\mathfrak{A}, \mathcal{M})$ is cohomologous to a cocycle σ which vanishes whenever any of its arguments lies in the centre $\mathcal{C}$ of $\mathfrak{A}$. In Section 4, we show (Theorem 4.1) that this remains true when $\mathcal{C}$ is replaced by certain non-central subalgebras of $\mathfrak{A}$. When $\mathfrak{A}$ is concretely represented and $\mathcal{M}$ is a dual normal $\mathfrak{A}^{-1}$-module, Theorem 4.1 can be strengthened (Lemma 5.4) with the additional conclusion that σ is (separately) ultraweakly continuous.

We recall some of the notation and terminology used in [7], [8]. With $\mathfrak{A}$ a Banach algebra and $\mathcal{M}$ a (two-sided) $\mathfrak{A}$-module, we describe $\mathcal{M}$ as a *Banach $\mathfrak{A}$-module* if the bilinear mappings $(A, m) \rightarrow A\,m$, $(A, m) \rightarrow m\,A$: $\mathfrak{A} \times \mathcal{M} \rightarrow \mathcal{M}$ are bounded. By a (continuous) *n-cochain*, we mean a bounded n-linear mapping from $\mathfrak{A} \times \mathfrak{A} \times \ldots \times \mathfrak{A}$ into $\mathcal{M}$, and we denote by $C_c^n(\mathfrak{A}, \mathcal{M})$ the linear space of all such n-cochains. The *coboundary operator*, from $C_c^n(\mathfrak{A}, \mathcal{M})$ into $C_c^{n+1}(\mathfrak{A}, \mathcal{M})$, is the linear operator Δ defined by

$$\begin{aligned}(\Delta\rho)(A_0, \ldots, A_n) &= A_0\,\rho(A_1, \ldots, A_n)\\ &+ \sum_{j=1}^{n}(-1)^j\,\rho(A_0, \ldots, A_{j-2}, A_{j-1}A_j, A_{j+1}, \ldots, A_n)\\ &+ (-1)^{n+1}\,\rho(A_0, \ldots, A_{n-1})\,A_n,\end{aligned}$$

for ρ in $C_c^n(\mathfrak{A}, \mathcal{M})$ and $A_0, \ldots, A_n$ in $\mathfrak{A}$. We adopt the convention that $C_c^0(\mathfrak{A}, \mathcal{M})$ is $\mathcal{M}$, and that $(\Delta\,m)(A) = A\,m - m\,A$ when $m \in \mathcal{M}$ and $A \in \mathfrak{A}$. For $n = 1, 2, \ldots$, we define

$$\begin{aligned}B_c^n(\mathfrak{A}, \mathcal{M}) &= \{\Delta\xi : \xi \in C_c^{n-1}(\mathfrak{A}, \mathcal{M})\},\\ Z_c^n(\mathfrak{A}, \mathcal{M}) &= \{\rho \in C_c^n(\mathfrak{A}, \mathcal{M}) : \Delta\rho = 0\}.\end{aligned}$$

Elements of the linear space $B_c^n(\mathfrak{A}, \mathcal{M})$ are called *n-coboundaries*, while the linear space $Z_c^n(\mathfrak{A}, \mathcal{M})$ consists of *n-cocycles*. Since $\Delta^2 = 0$, we have $B_c^n(\mathfrak{A}, \mathcal{M}) \subseteq Z_c^n(\mathfrak{A}, \mathcal{M})$; the quotient space

$$H_c^n(\mathfrak{A}, \mathcal{M}) = Z_c^n(\mathfrak{A}, \mathcal{M})/B_c^n(\mathfrak{A}, \mathcal{M})$$

is called the *n-dimensional* (continuous) *cohomology group* of $\mathfrak{A}$, with coefficients in $\mathcal{M}$.

With $\mathfrak{A}$ a Banach algebra and $\mathcal{M}$ a Banach $\mathfrak{A}$-module, we describe $\mathcal{M}$ as a *dual* $\mathfrak{A}$*-module* if $\mathcal{M}$ is (isometrically isomorphic to) the dual space of a Banach space $\mathcal{M}_*$ and, for each A in $\mathfrak{A}$, the mappings $m \to A\, m$, $m \to m\, A : \mathcal{M} \to \mathcal{M}$ are weak * continuous. If, further, $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$ and, for each m in $\mathcal{M}$, the mappings $A \to A\, m$, $A \to m\, A : \mathfrak{A} \to \mathcal{M}$ are ultraweak-weak*continuous, we describe $\mathcal{M}$ as a *dual normal* $\mathfrak{A}$*-module* (the simplest example is obtained by taking $\mathcal{M} = \mathfrak{A}^-$, the ultraweak closure of $\mathfrak{A}$). In the context of dual normal modules, we denote by $C^n_w(\mathfrak{A}, \mathcal{M})$ the set of all elements ρ of $C^n_c(\mathfrak{A}, \mathcal{M})$ which are separately ultraweak-weak*continuous [with $C^0_w(\mathfrak{A}, \mathcal{M}) = \mathcal{M}$], and observe that Δ maps $C^n_w(\mathfrak{A}, \mathcal{M})$ into $C^{n+1}_w(\mathfrak{A}, \mathcal{M})$. We define

$$B^n_w(\mathfrak{A}, \mathcal{M}) = \{ \Delta\xi : \xi \in C^{n-1}_w(\mathfrak{A}, \mathcal{M}) \},$$
$$Z^n_w(\mathfrak{A}, \mathcal{M}) = \{ \rho \in C^n_w(\mathfrak{A}, \mathcal{M}) : \Delta\rho = 0 \},$$
$$H^n_w(\mathfrak{A}, \mathcal{M}) = Z^n_w(\mathfrak{A}, \mathcal{M})/B^n_w(\mathfrak{A}, \mathcal{M}).$$

In this context, we refer to *normal* n-cochains, coboundaries, cocycles, and we call $H^n_w(\mathfrak{A}, \mathcal{M})$ the n-dimensional *normal* cohomology group.

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2. Extensions of ultraweakly continuous multilinear mappings

After two preparatory lemmas, we prove the main result of this section (Theorem 2.3), concerning the extension of n-linear mappings. When X and Y are Banach spaces in duality, we denote by $\sigma(X, Y)$ the weak topology induced on X by Y.

Lemma 2.1. — *If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras acting on Hilbert spaces $\mathcal{H}_{\mathfrak{A}}$, $\mathcal{H}_{\mathfrak{B}}$ respectively, and τ is a bounded bilinear functional on $\mathfrak{A} \times \mathfrak{B}$ which is separately ultraweakly continuous, then τ extends uniquely, without change of norm, to a bounded bilinear functional τ_1 on $\mathfrak{A} \times \mathfrak{B}^-$ which is separately ultraweakly continuous.*

Proof. — For each A in $\mathfrak{A}$, the mapping $B \to \tau(A, B)$ is an ultraweakly continuous linear functional $S(A)$ on $\mathfrak{B}$, and $\| S(A) \| \leq \| \tau \| \| A \|$. By the Kaplansky density theorem, $S(A)$ extends without change of norm to an ultraweakly continuous linear functional $T(A)$ on $\mathfrak{B}^-$. Thus, T is a bounded linear mapping from $\mathfrak{A}$ into the predual $(\mathfrak{B}^-)_*$ of $\mathfrak{B}^-$, and $\| T \| \leqq \| \tau \|$. Moreover,

$$\tau(A, B) = \langle T(A), B \rangle \qquad (A \in \mathfrak{A}, B \in \mathfrak{B}),$$

where $\langle \ , \ \rangle$ denotes the canonical bilinear functional arising from the duality between $(\mathfrak{B}^-)_*$ and $\mathfrak{B}^-$. Since τ is ultraweakly continuous in its first argument, for each fixed B in $\mathfrak{B}$, T is continuous as a mapping from $\mathfrak{A}$ [with topology $\sigma(\mathfrak{A}, (\mathfrak{A}^-)_*)$] into $(\mathfrak{B}^-)_*$ [with topology $\sigma((\mathfrak{B}^-)_*, \mathfrak{B})$].

With $\mathfrak{A}_1$ the unit ball of $\mathfrak{A}$, it follows from [1] (Corollary II.9) that $T(\mathfrak{A}_1)$ is relatively compact in the topology $\sigma((\mathfrak{B}^-)_*, \mathfrak{B}^-)$; so this topology coincides, on $T(\mathfrak{A}_1)$, with the coarser Hausdorff topology $\sigma((\mathfrak{B}^-)_*, \mathfrak{B})$. This, together with the final statement in the preceding paragraph, shows that T is continuous as a mapping from $\mathfrak{A}_1$, with the topology $\sigma(\mathfrak{A}, (\mathfrak{A}^-)_*)$, into $(\mathfrak{B}^-)_*$, with the topology $\sigma((\mathfrak{B}^-)_*, \mathfrak{B}^-)$.

With τ_1 defined by

$$\tau_1(A, B) = \langle T(A), B \rangle \qquad (A \in \mathfrak{A}, B \in \mathfrak{B}^-),$$

τ_1 is a bounded bilinear functional on $\mathfrak{A} \times \mathfrak{B}^-$, extends τ, and satisfies $\| \tau_1 \| \leqq \| T \| \leqq \| \tau \|$ (whence, $\| \tau_1 \| = \| \tau \|$). For each B in $\mathfrak{B}^-$, the linear functional $A \to \tau_1(A, B)$ on $\mathfrak{A}$ is ultraweakly continuous on $\mathfrak{A}_1$ (hence on $\mathfrak{A}$ [3], Théorème 1, p. 38), by the continuity property of T established in the preceding paragraph. Ultraweak continuity of τ_1 in its second argument, for each fixed A in $\mathfrak{A}$, is apparent since $T(A) \in (\mathfrak{B}^-)_*$. The uniqueness of such a bilinear functional τ_1 is an immediate consequence of this continuity.

LEMMA 2.2. — *If $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras acting on Hilbert spaces $\mathcal{H}_{\mathfrak{A}}$, $\mathcal{H}_{\mathfrak{B}}$ respectively, $\mathcal{M}$ is the dual space of a complex Banach space $\mathcal{M}_*$, and $\rho : \mathfrak{A} \times \mathfrak{B} \to \mathcal{M}$ is a bounded bilinear mapping which is separately ultraweak-weak * continuous, then ρ extends uniquely, without change of norm, to a bounded bilinear mapping $\bar{\rho} : \mathfrak{A} \times \mathfrak{B}^- \to \mathcal{M}$ which is separately ultraweak-weak * continuous.*

Proof. — When $m \in \mathcal{M}$ and $\omega \in \mathcal{M}_*$, we write $\langle m, \omega \rangle$ in place of $m(\omega)$. With ω in $\mathcal{M}_*$ and l_ω defined by

$$(1) \qquad l_\omega(A, B) = \langle \rho(A, B), \omega \rangle \qquad (A \in \mathfrak{A}, B \in \mathfrak{B}),$$

l_ω is a bounded bilinear functional on $\mathfrak{A} \times \mathfrak{B}^-$, with $\| l_\omega \| \leqq \| \omega \| . \| \rho \|$, and is separately ultraweakly continuous. By the preceding lemma, l_ω extends uniquely, without change of norm, to a bounded bilinear functional L_ω on $\mathfrak{A}\times\mathfrak{B}^-$, which is again separately ultraweakly continuous. For each A in $\mathfrak{A}$ and B in $\mathfrak{B}^-$, the mapping $\bar\rho\,(A, B) : \omega \to L_\omega(A, B)$ is a bounded linear functional on $\mathcal{M}_*$ (that is, an element of $\mathcal{M}$), and $\|\bar\rho\,(A, B)\| \leq \|\rho\|.\|A\|.\|B\|$. Since

$$(2) \qquad \langle \bar\rho\,(A, B), \omega \rangle = L_\omega\,(A, B) \qquad (\omega\in\mathcal{M}_*,\ A\in\mathfrak{A},\ B\in\mathfrak{B}^-),$$

it is clear that $\bar\rho$ is a bounded bilinear mapping from $\mathfrak{A}\times\mathfrak{B}^-$ into $\mathcal{M}$, with $\|\bar\rho\| \leq \|\rho\|$, and is separately ultraweak-weak * continuous. Since L_ω extends l_ω, it follows from (1) and (2) that $\bar\rho$ extends ρ (whence, $\|\bar\rho\| = \|\rho\|$). The uniqueness of this extension follows by ultraweak-weak * continuity of $\bar\rho$ in its second argument.

THEOREM 2.3. — *If $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$ are C*-algebras acting on Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ respectively, $\mathcal{M}$ is the dual space of a complex Banach space $\mathcal{M}_*$, and $\rho : \mathfrak{A}_1\times\ldots\times\mathfrak{A}_n \to \mathcal{M}$ is a bounded multilinear mapping which is separately ultraweak-weak * continuous, then ρ extends uniquely, without change of norm, to a bounded multilinear mapping $\bar\rho$: $\mathfrak{A}_1^-\times\ldots\times\mathfrak{A}_n^- \to \mathcal{M}$ which is separately ultraweak-weak * continuous.*

Proof. — We construct, in succession, multilinear mappings $\rho_0\,(=\rho)$, ρ_1, ρ_2, $\ldots$, ρ_n with the following properties : ρ_k maps $\mathfrak{A}_1^-\times\ldots\times\mathfrak{A}_k^-\times\mathfrak{A}_{k+1}\times\ldots\times\mathfrak{A}_n$ into $\mathcal{M}$, extends ρ_{k-1} without change of norm when $k \geq 1$, and is separately ultraweak-weak * continuous. This proves the existence of a suitable $\bar\rho\,(=\rho_n)$; and its uniqueness results from the stated continuity properties.

Suppose that $1 \leq j \leq n$, and suitable $\rho_0, \ldots, \rho_{j-1}$ have been constructed. For each fixed A_1 in $\mathfrak{A}_1^-, \ldots, A_{j-1}$ in $\mathfrak{A}_{j-1}^-$, A_{j+1} in $\mathfrak{A}_{j+1}, \ldots, A_n$ in $\mathfrak{A}_n$, the mapping $A_j \to \rho_{j-1}\,(A_1, \ldots, A_n)$ of $\mathfrak{A}_j$ into $\mathcal{M}$ is ultraweak-weak * continuous, and has norm not greater than

$$\|\rho\|.\|A_1\| \ldots \|A_{j-1}\|.\|A_{j+1}\| \ldots \|A_n\|.$$

By weak * completeness of the unit ball in $\mathcal{M}$, together with the Kaplansky density theorem, it extends without increase of norm to an ultraweak-weak * continuous linear mapping $A_j \to \rho_j\,(A_1, \ldots, A_n)$ of $\mathfrak{A}_j^-$ into $\mathcal{M}$. It is clear that ρ_j is an extension of ρ_{j-1} to a bounded multilinear mapping from $\mathfrak{A}_1^-\times\ldots\times\mathfrak{A}_j^-\times\mathfrak{A}_{j+1}\times\ldots\times\mathfrak{A}_n$, with $\|\rho_j\| = \|\rho_{j-1}\|$, and that ρ_j is ultraweak-weak * continuous in its j-th argument. It remains to prove the same continuity in the remaining arguments.

To simplify the notation, let $\mathfrak{B}_i$ denote $\mathfrak{A}_i^-$ when $1 \leq i < j$, $\mathfrak{A}_i$ when $j < i \leq n$. With $1 \leq k \leq n$ and $k \neq j$, choose and fix A_i in $\mathfrak{B}_i$ for each $i = 1, \ldots, n$ other than j, k. The bounded bilinear mapping σ : $\mathfrak{A}_j\times\mathfrak{B}_k \to \mathcal{M}$, defined by

$$(3) \qquad \sigma\,(A_j, A_k) = \rho_{j-1}\,(A_1, \ldots, A_n) = \rho_j\,(A_1, \ldots, A_n),$$

is separately ultraweak-weak * continuous, by our assumptions concerning ρ_{j-1}. By Lemma 2.2, σ extends to a bounded bilinear mapping $\bar{\sigma}$: $\mathfrak{A}_j^- \times \mathfrak{B}_k \to \mathcal{M}$, which is again separately ultraweak-weak * continuous. Using this continuity, of both ρ_j and $\bar{\sigma}$ in the variable A_j, we deduce from (3) that

$$\bar{\sigma}(A_j, A_k) = \rho_j(A_1, \ldots, A_n) \qquad (A_j \in \mathfrak{A}_j^-, A_k \in \mathfrak{B}_k).$$

Since the left hand side is ultraweak-weak * continuous in A_k, the same is true of $\rho_j(A_1, \ldots, A_n)$.

3. Ultraweak continuity and complete additivity of linear mappings

In [9] (Corollary 1), Takesaki exploits the properties of the universal representation of a von Neumann algebra $\mathcal{R}$ to characterise the ultraweakly continuous linear functionals on $\mathcal{R}$ as those which are completely additive (on families of orthogonal projections). We shall need this result — or rather, an immediate consequence of it (Corollary 3.4) characterising ultraweakly continuous linear mappings between von Neumann algebras — in Section 5. It is surprising, at first glance, that this basic result does not follow easily from the corresponding fact for positive functionals. It seems worthwhile to have a proof entirely within the framework of von Neumann algebras. We give such a proof; and, for completeness, we include an account of (the essence of) Takesaki's original argument.

Suppose that $\mathcal{R}$ and $\mathcal{S}$ are von Neumann algebras, ω is a bounded linear functional on $\mathcal{R}$, and ξ is a bounded linear mapping from $\mathcal{R}$ into $\mathcal{S}$. We say that ω is *completely additive* if $\omega\left(\sum E_\alpha\right) = \sum \omega(E_\alpha)$ for every orthogonal family (E_α) of projections in $\mathcal{R}$. Similarly, ξ is *completely additive* if $\sum \xi(E_\alpha)$ converges ultraweakly to $\xi\left(\sum E_\alpha\right)$, for each such family (E_α). It is clear that ultraweak continuity, of ω or ξ, implies complete additivity; the main results in this section establish the equivalence of the two conditions. Before proving these results, we require some lemmas.

Lemma 3.1. — *With E an (orthogonal) projection, distinct from 0 and I, on the Hilbert space $\mathcal{H}$ and a, b, c real numbers,*

$$aE + b(I - E) + c[E\,T(I - E) + (I - E)\,T^* E] \geq 0$$

for each T in the unit ball of $\mathcal{B}(\mathcal{H})$, if and only if a and b are non-negative, and $ab \geq c^2$.

Proof. — Given a, b and $ab - c^2$ are non-negative, and $\| T \| \leq 1$,

$$\begin{aligned} a \langle E x, x \rangle &+ b \langle (I - E) x, x \rangle \\ &+ c \langle ET (I - E) x, x \rangle + c \langle (I - E) T^* E x, x \rangle \\ &= a \| E x \|^2 + b \| (I - E) x \|^2 \\ &\quad + c [\langle T (I - E) x, E x \rangle + \langle T^* E x, (I - E) x \rangle] \\ &\geq a \| E x \|^2 + b \| (I - E) x \|^2 \\ &\quad - 2 | c | \, \| (I - E) x \| . \| E x \| \geq 0, \end{aligned}$$

for each x in $\mathcal{H}$; since the quadratic form $as^2 + bt^2 - 2 | c | st$ is positive semi-definite.

Conversely, suppose

$$a E + b (I - E) + c [ET (I - E) + (I - E) T^* E] \geq 0$$

with T^* a partial isometry having initial and final projections one-dimensional, dominated by E and $I - E$ respectively. Restriction to the two-dimensional subspace generated by the corresponding one-dimensional subspaces yields a positive operator H. Considering the matrix of H (relative to the obvious orthonormal basis), we conclude that a and b are non-negative and $ab \geq c^2$.

LEMMA 3.2. — *Suppose that ω is a bounded hermitian linear functional on a von Neumann algebra $\mathcal{R}$, and η is a real number.*

(i) *If $\omega (A) > \eta$ for some positive A in the unit ball of $\mathcal{R}$, then there is a projection E in $\mathcal{R}$ such that $\omega (E) > \eta$; moreover, E can be chosen so that $\omega \mid E \mathcal{R} E$ is a positive linear functional if ω is completely additive.*

(ii) *If $| \omega (F) | \leq \eta$, for every projection F in $\mathcal{R}$, then $\| \omega \| \leq 4 \eta$.*

Proof. —

(i) By the spectral theorem, there exists an orthogonal family $(E_1, \ldots, E_n)$ of projections in $\mathcal{R}$, and scalars $\lambda_1, \ldots, \lambda_n$ in $(0, 1]$, such that $\| A - \sum \lambda_j E_j \| < \| \omega \|^{-1} (\omega (A) - \eta)$. Thus

$$\omega (A) - \sum_{j=1}^{n} \lambda_j \, \omega (E_j) < \omega (A) - \eta, \qquad \sum_{j=1}^{n} \lambda_j \, \omega (E_j) > \eta.$$

Renumbering if necessary, we may suppose that $\omega (E_j) > 0$ $(1 \leq j \leq m)$ and $\omega (E_j) \leq 0$ $(m < j \leq n)$, for some m with $0 \leq m \leq n$. With

$$E = \sum_{j=1}^{m} E_j,$$

$$\omega (E) = \sum_{j=1}^{m} \omega (E_j) \geq \sum_{j=1}^{m} \lambda_j \, \omega (E_j) \geq \sum_{j=1}^{n} \lambda_j \, \omega (E_j) > \eta.$$

If ω is completely additive, let (F_α) be a maximal orthogonal family of subprojections of E in $\mathcal{R}$ such that $\omega(F_\alpha) < 0$; and let $E_0 = E - \sum F_\alpha$. By the maximality assumption, $\omega(F) \geq 0$ for every subprojection F of E_0 in $\mathcal{R}$, so $\omega \mid E_0 \mathcal{R} E_0$ is a positive linear functional. Furthermore,

$$\omega(E_0) = \omega(E) - \omega\left(\sum F_\alpha\right) = \omega(E) - \sum \omega(F_\alpha) \geq \omega(E) > \eta.$$

(ii) If $|\omega(F)| \leq \eta$, for every projection F in $\mathcal{R}$, it follows, by applying part (i) to both ω and $-\omega$, that $|\omega(A)| \leq \eta$ for every positive A in the unit ball $\mathcal{R}_1$ of $\mathcal{R}$. Each R in $\mathcal{R}_1$ has the form $R = A_1 - A_2 + i(A_3 - A_4)$, with A_j a positive element of $\mathcal{R}_1$, and

$$|\omega(R)| \leq \sum_{j=1}^{4} |\omega(A_j)| \leq 4\eta.$$

THEOREM 3.3. — *Each bounded completely additive linear functional ω on a von Neumann algebra $\mathcal{R}$ is ultraweakly continuous.*

Proof. — We recall that the ultraweakly continuous linear functionals on $\mathcal{R}$ form a norm closed subspace $\mathcal{R}_*$ of the Banach dual space $\mathcal{R}^*$ of $\mathcal{R}$, and that a positive linear functional lies in $\mathcal{R}_*$ if and only if it is completely additive ([3], Théorème 1, (iii), p. 38; Exercice 9, p. 68). Given any completely additive element ω of $\mathcal{R}^*$ (not necessarily positive), we shall prove that $\omega \in \mathcal{R}_*$ by showing that ω can be approximated in norm by elements of $\mathcal{R}_*$. Since the hermitian and skew hermitian parts of ω are completely additive, we may suppose, without loss of generality, that ω is hermitian and $\|\omega\| \leq 1$.

If

$$\mu = \sup\{\omega(A) : A = A^* \in \mathcal{R}, 0 \leq A \leq I\}, \tag{4}$$

then $0 \leq \mu \leq \|\omega\| \leq 1$. Given ε such that $0 < \varepsilon \leq \frac{3}{4}$, there is a positive operator E_1 in the unit ball of $\mathcal{R}$ such that $\omega(E_1) > \mu - \varepsilon$. By Lemma 3.2 (i), we may suppose that E_1 is a projection, and $\omega \mid E_1 \mathcal{R} E_1$ is a positive linear functional. Since $\omega \mid E_1 \mathcal{R} E_1$ is also completely additive, it is ultraweakly continuous. With $E_2 = I - E_1$, ω is the sum of the four linear functionals $\omega_{j,k}$ $(j, k = 1, 2)$, defined by $\omega_{j,k}(R) = \omega(E_j R E_k)$, and $\omega_{1,1} \in \mathcal{R}_*$. Note also that, if F is any projection in $\mathcal{R}$ satisfying $F \leq E_2$, then, by (4),

$$\omega(F) = \omega(E_1 + F) - \omega(E_1) < \mu - (\mu - \varepsilon) = \varepsilon.$$

We show next that $\|\omega_{1,2}\|$ and $\|\omega_{2,1}\|$ are small. For this, suppose that T is in the unit ball of $\mathcal{R}$, and let

$$S = (1-\varepsilon)E_1 + \varepsilon E_2 + \varepsilon^{\frac{1}{2}}(1-\varepsilon)^{\frac{1}{2}}(E_1 T E_2 + E_2 T^* E_1),$$

so that

$$I - S = \varepsilon E_1 + (1 - \varepsilon) E_2 - \varepsilon^{\frac{1}{2}} (1 - \varepsilon)^{\frac{1}{2}} (E_1 TE_2 + E_2 T^* E_1).$$

By Lemma 3.1, both S and $I - S$ are positive; so $0 \leqq S \leqq I$, and (4) implies that

$$\begin{aligned}\mu \geqq \omega (S) &= (1 - \varepsilon) \omega (E_1) + \varepsilon\omega (E_2) + \varepsilon^{\frac{1}{2}} (1 - \varepsilon)^{\frac{1}{2}} \omega (E_1 TE_2 + E_2 T^* E_1) \\ &\geqq (1 - \varepsilon) (\mu - \varepsilon) - \varepsilon + 2 \varepsilon^{\frac{1}{2}} (1 - \varepsilon)^{\frac{1}{2}} \operatorname{Re} \omega (E_1 TE_2) \\ &\geqq \mu - (\mu + 2) \varepsilon + 2 \varepsilon^{\frac{1}{2}} (1 - \varepsilon)^{\frac{1}{2}} \operatorname{Re} \omega (E_1 TE_2).\end{aligned}$$

Since $\mu \leqq 1$ and $0 < \varepsilon \leqq \frac{3}{4}$,

$$\begin{aligned}\operatorname{Re} \omega_{1,2} (T) &= \operatorname{Re} \omega (E_1 TE_2) \\ &\leqq \frac{1}{2}(\mu + 2) \varepsilon^{\frac{1}{2}} (1 - \varepsilon)^{-\frac{1}{2}} \leqq 3 \sqrt{\varepsilon}.\end{aligned}$$

This last inequality is valid for all T in the unit ball of $\mathcal{R}$; so $\| \omega_{1,2} \| \leqq 3 \sqrt{\varepsilon}$, and similarly $\| \omega_{2,1} \| \leqq 3 \sqrt{\varepsilon}$. Thus

$$\| \omega - \omega_{1,1} - \omega_{2,2} \| = \| \omega_{1,2} + \omega_{2,1} \| \leqq 6 \sqrt{\varepsilon}. \tag{5}$$

We prove next that $\| \omega_{2,2} + \omega_0 \| \leqq 4 \varepsilon + 6 \sqrt{\varepsilon}$, for some ω_0 in $\mathcal{R}_*$. For this, we consider the restriction $\nu = - \omega \mid E_2 \mathcal{R} E_2$, which is a completely additive linear functional on $E_2 \mathcal{R} E_2$ satisfying $\| \nu \| \leqq 1$ and $\nu (F) > - \varepsilon$ for each projection F in $E_2 \mathcal{R} E_2$. By applying to ν the argument used above for ω, we deduce the existence of projections F_1 and F_2 in $E_2 \mathcal{R} E_2$, with sum E_2, satisfying the following conditions : if, for $j, k = 1, 2$, a linear functional $\nu_{j,k}$ on $E_2 \mathcal{R} E_2$ is defined by $\nu_{j,k} (S) = \nu (F_j SF_k)$ $(S \in E_2 \mathcal{R} E_2)$, then $\nu_{1,1}$ is ultraweakly continuous,

$$\| \nu - \nu_{1,1} - \nu_{2,2} \| \leqq 6 \sqrt{\varepsilon} \tag{6}$$

and $\nu (F) < \varepsilon$ for each projection F in $E_2 \mathcal{R} E_2$ such that $F \leqq F_2$. This last inequality, together with our previous result in the reverse direction, shows that $| \nu (F) | < \varepsilon$ for each projection F in $F_2 \mathcal{R} F_2$. By Lemma 3.2 (ii), $\| \nu \mid F_2 \mathcal{R} F_2 \| \leqq 4 \varepsilon$; whence

$$| \nu_{2,2} (S) | = | \nu (F_2 SF_2) | \leqq 4 \varepsilon \| F_2 SF_2 \| \leqq 4 \varepsilon \| S \|$$

$(S \in E_2 \mathcal{R} E_2)$, and $\| \nu_{2,2} \| \leqq 4\varepsilon$. This, with (6), yields

$$\| \nu - \nu_{1,1} \| \leqq 4 \varepsilon + 6 \sqrt{\varepsilon}.$$

With ω_0, defined by $\omega_0(R) = \nu_{1,1}(E_2 R E_2)$ for R in $\mathcal{R}$, we have $\omega_0 \in \mathcal{R}_*$ and

$$\begin{aligned} |(\omega_0 + \omega_{2,2})(R)| &= |\nu_{1,1}(E_2 R E_2) + \omega(E_2 R E_2)| \\ &= |(\nu_{1,1} - \nu)(E_2 R E_2)| \\ &\leq \|\nu_{1,1} - \nu\| . \|E_2 R E_2\| \\ &\leq (4\varepsilon + 6\sqrt{\varepsilon})\|R\| \qquad (R \in \mathcal{R}). \end{aligned}$$

Thus $\|\omega_0 + \omega_{2,2}\| \leq 4\varepsilon + 6\sqrt{\varepsilon}$ and, by (5),

$$\|\omega - \omega_{1,1} + \omega_0\| \leq 4\varepsilon + 12\sqrt{\varepsilon}.$$

Since $\mathcal{R}_*$ is closed, and the above construction of ω_0, $\omega_{1,1}$ in $\mathcal{R}_*$ is possible for each positive ε, it follows that $\omega \in \mathcal{R}_*$.

COROLLARY 3.4. — *Each bounded completely additive linear mapping ξ, from a von Neumann algebra $\mathcal{R}$ into another such algebra $\mathcal{S}$, is ultraweakly continuous.*

Proof. — Suppose $\omega \in \mathcal{S}_*$. For each orthogonal family (E_α) of projections in $\mathcal{R}$, $\sum \xi(E_\alpha)$ converges ultraweakly to $\xi\left(\sum E_\alpha\right)$, and thus

$$\omega\left(\xi\left(\sum E_\alpha\right)\right) = \sum \omega(\xi(E_\alpha)).$$

It follows that the linear functional $\omega \circ \xi : A \to \omega(\xi(A))$ on $\mathcal{R}$ is completely additive. By Theorem 3.3, $\omega \circ \xi$ is ultraweakly continuous. Since this is so for each ω in $\mathcal{S}_*$, ξ is ultraweakly continuous.

REMARK 3.5. — Suppose that ω is a bounded linear functional on a von Neumann algebra $\mathcal{R}$, whose restriction to each maximal abelian subalgebra of $\mathcal{R}$ is ultraweakly continuous. With (E_α) an orthogonal family of projections on $\mathcal{R}$, there is a maximal abelian sublgebra $\mathcal{A}$ of $\mathcal{R}$ which contains each E_α. The ultraweak continuity of the restriction $\omega \mid \mathcal{A}$ implies that $\omega\left(\sum E_\alpha\right) = \sum \omega(E_\alpha)$; so ω is completely additive. By Theorem 3.3, ω is ultraweakly continuous on $\mathcal{R}$. Thus Theorem 3.3 implies the following result of TAKESAKI ([9], Corollary 1) (and is, essentially, equivalent to it). We conclude this section with a second proof, closely parallel in its broad outline with the ideas used in [9].

THEOREM 3.6. (TAKESAKI). — *A bounded linear functional ω on a von Neumann algebra $\mathcal{R}$, whose restriction to each maximal abelian sulalgebra of $\mathcal{R}$ is ultraweakly continuous, is ultraweakly continuous on $\mathcal{R}$.*

Proof. — Since it suffices to prove the result for any von Neumann algebra isomorphic to $\mathcal{R}$, we may assume (just as in the proof of [8], Theorem 2.1) that $\mathcal{R} = \mathcal{R}_0 P$, where $\mathcal{R}_0$ acting on $\mathcal{H}_0$ is the universal representation of $\mathcal{R}$, and P is a central projection in $\mathcal{R}_0^-$. Since $\mathcal{R}$ is ultraweakly closed, $\mathcal{R} = \mathcal{R}_0 P = \mathcal{R}_0^- P$.

When we refer to the ultraweak topology on $\mathcal{R}_0$ or $\mathcal{R}_0^-$, we mean the one arising from the action of those algebras on $\mathcal{H}_0$. By the ultraweak topology on $\mathcal{R}$ $(= \mathcal{R}_0^- P \subseteq \mathcal{R}_0^-)$, we mean the one arising from the action of $\mathcal{R}$ on $P(\mathcal{H}_0)$, and this coincides with the restriction to $\mathcal{R}$ of the ultraweak topology on $\mathcal{R}_0^-$.

With f a bounded linear functional on $\mathcal{R}$, we denote by f_P the bounded linear functional $A \to f(AP)$ on $\mathcal{R}_0$, and by $\bar{f}_P$ the extension of f_P to an ultraweakly continuous linear functional on $\mathcal{R}_0^-$. Since $\mathcal{R} = \mathcal{R}_0^- P \subseteq \mathcal{R}_0^-$, the restriction $\bar{f}_P \mid \mathcal{R}$ is an ultraweakly continuous functional on $\mathcal{R}$.

If f is ultraweakly continuous, then so is the linear functional g : $A \to f(AP)$ on $\mathcal{R}_0^-$. Since g and $\bar{f}_P$ have the same restriction, f_P, to $\mathcal{R}_0$, their ultraweak continuity entails $g = \bar{f}_P$. Thus

$$\bar{f}_P(AP) = g(AP) = f(AP) \qquad (A \in \mathcal{R}_0^-),$$

and so $f = \bar{f}_P \mid \mathcal{R}$. This, with the preceding paragraph, shows that f is ultraweakly continuous if and only if $f = \bar{f}_P \mid \mathcal{R}$.

Accordingly, we have to show that $\omega = \bar{\omega}_P \mid \mathcal{R}$. As in the proof of Theorem 3.3, we may assume that ω is Hermitian, and the same is then true of $\omega - (\bar{\omega}_P \mid \mathcal{R})$ $(= g)$. Furthermore,

$$\begin{aligned} (7) \qquad g(AP) &= \omega(AP) - \bar{\omega}_P(AP) \\ &= \omega_P(A) - \bar{\omega}_P(AP) = \bar{\omega}_P(A - AP) \qquad (A \in \mathcal{R}_0). \end{aligned}$$

Since ω and $\bar{\omega}_P \mid \mathcal{R}$ are completely additive ($\bar{\omega}_P \mid \mathcal{R}$, because it is ultraweakly continuous), the same is true of g.

If $g \neq 0$, then $g(E_0) \neq 0$ for some projection E_0 in $\mathcal{R}$; we may assume $g(E_0) > 0$. The " exhaustion " argument, used during the proof of Lemma 3.2 (i), now shows that $g \mid E_1 \mathcal{R} E_1$ is a non-zero *positive* linear functional (and, of course, completely additive), for some non-zero subprojection E_1 of E_0 in $\mathcal{R}$. It follows that $g \mid E_1 \mathcal{R} E_1$ is ultraweakly continuous (see the first sentence of the proof of Theorem 3.3). Hence the linear functional $h : A \to g(E_1 A E_1)$ on $\mathcal{R}$ is non-zero and ultraweakly continuous (whence, $h = \bar{h}_P \mid \mathcal{R}$). By (7),

$$\bar{h}_P(A) = h_P(A) = h(AP) = g(E_1 APE_1) = \bar{\omega}_P(E_1 AE_1 (I - P)),$$

for each A in $\mathcal{R}_0$. By the ultraweak continuity of $\bar{h}_P$ and $\bar{\omega}_P$, we have $\bar{h}_P(A) = \bar{\omega}_P(E_1 AE_1(I - P))$, for all A in $\mathcal{R}_0^-$. Thus $\bar{h}_P(AP) = 0$

when $A \in \mathcal{R}_0^-$. It follows that

$$h = \bar{h}_P \mid \mathcal{R} = \bar{h}_P \mid \mathcal{R}_0^- P = 0,$$

contradicting our earlier conclusion that $h \neq 0$. Thus $g = 0$ and $\omega = \bar{\omega}_P \mid \mathcal{R}$.

4. Adjustment of cocycles relative to an amenable subalgebra

With $\mathfrak{A}$ a C^*-algebra and $\mathcal{M}$ a two-sided $\mathfrak{A}$-module, [7] (Theorem 3.4) asserts that each ρ in $Z_c^n(\mathfrak{A}, \mathcal{M})$ is cohomologous to a cocycle which vanishes whenever any of its arguments lies in the centre $\mathcal{C}$ of $\mathfrak{A}$. In this section, we obtain a stronger result of the same type (Theorem 4.1), in which $\mathfrak{A}$ is a Banach algebra and $\mathcal{C}$ is a closed subalgebra (not necessarily central) which is amenable in the sense of [5] (Section 5). Before proving this theorem, we recall and slightly augment some results from [5].

Let $\mathfrak{A}$ be a Banach algebra. Elsewhere, in [7], [8] and the present paper, we assume for simplicity that our $\mathfrak{A}$-modules $\mathcal{M}$ are *unital* ($1\,m = m\,1 = m$ for each m in $\mathcal{M}$), when $\mathfrak{A}$ has an identity element 1. *This assumption is not in force in the present section* : the discussion in [5] (Section 1 (*c*)) shows that the choice between using unital or more general modules is a matter of minor convenience only. Suppose that $\mathcal{M}$ is a two-sided dual $\mathfrak{A}$-module, so that $\mathcal{M}$ is (isometrically isomorphic to) the dual space of a Banach space $\mathcal{M}_*$, and the linear mappings $m \to A\,m$, $m \to m\,A$: $\mathcal{M} \to \mathcal{M}$ are weak * continuous, for each fixed A in $\mathfrak{A}$. In view of this continuity, these mappings are the adjoints of certain bounded linear operators acting on $\mathcal{M}_*$, which we denote by $\omega \to \omega\,A$, $\omega \to A\,\omega$, respectively. In this way, $\mathcal{M}_*$ acquires the structure of a Banach $\mathfrak{A}$-module. Thus the class of " dual $\mathfrak{A}$-modules ", considered in [7], [8], coincides with the class $\{ \mathfrak{X}^* : \mathfrak{X}$ is a Banach $\mathfrak{A}$-module $\}$, used in [5].

We denote by $\langle\, ,\rangle$ the bilinear functional on $\mathcal{M} \times \mathcal{M}_*$, arising from the duality between $\mathcal{M}$ and $\mathcal{M}_*$. With p a positive integer, $C_c^p(\mathfrak{A}, \mathcal{M})$ is isometrically isomorphic to the dual space of the projective tensor product $\mathfrak{A} \hat{\otimes} \mathfrak{A} \hat{\otimes} \ldots \hat{\otimes} \mathfrak{A} \hat{\otimes} \mathcal{M}_*$, an element ξ of $C_c^p(\mathfrak{A}, \mathcal{M})$ corresponding to the linear functional $\bar{\xi}$, defined by

$$\bar{\xi}(A_1 \otimes \ldots \otimes A_p \otimes \omega) = \langle \xi(A_1, \ldots, A_p), \omega \rangle,$$

for all $A_1, \ldots, A_p$ in $\mathfrak{A}$ and ω in $\mathcal{M}_*$. We recall, from [5] (Section 1 (*a*)) that $C_c^p(\mathfrak{A}, \mathcal{M})$ has a dual $\mathfrak{A}$-module structure defined by

$$(8)\begin{cases} (A_0\,\xi)(A_1, \ldots, A_p) = A_0\,\xi(A_1, \ldots, A_p), \\ (\xi\,A_0)(A_1, \ldots, A_p) = \sum_{j=0}^{p-1} (-1)^j\,\xi(A_0, \ldots, A_{j-1}, A_j\,A_{j+1}, A_{j+2}, \ldots, A_p) \\ \qquad + (-1)^p\,\xi(A_0, \ldots, A_{p-1})\,A_p. \end{cases}$$

Furthermore, $H_c^{n+p}(\mathfrak{A}, \mathcal{M}) \simeq H_c^n(\mathfrak{A}, C_c^p(\mathfrak{A}, \mathcal{M}))$, $n = 1, 2, \ldots$. The ideas just described are analogous to methods, used by HOCHSCHILD ([4], Section 3), in the purely algebraic setting.

We recall, from [5] (Section 5), that a Banach algebra $\mathfrak{A}$ is said to be amenable if $H_c^1(\mathfrak{A}, \mathcal{M}) = 0$, for every two-sided dual $\mathfrak{A}$-module $\mathcal{M}$. This condition entails $H_c^n(\mathfrak{A}, \mathcal{M}) = 0$ $(n = 1, 2, \ldots)$, for each such $\mathcal{M}$, in view of the discussion in the preceding paragraph. Postliminal C^*-algebras (in particular, abelian ones) and uniformly hyperfinite C^*-algebras are amenable ([5], Theorem 7.9, and remarks, following the proof of Lemma 7.13; [8], Corollary 3.4).

With $\mathfrak{A}$ a Banach algebra, $\mathcal{M}$ a two-sided dual $\mathfrak{A}$-module, and k a positive integer, $C_c^k(\mathfrak{A}, \mathcal{M})$ can (as usual) be identified with the dual space of $\mathfrak{A} \hat{\otimes} \mathfrak{A} \hat{\otimes} \ldots \hat{\otimes} \mathfrak{A} \hat{\otimes} \mathcal{M}_*$, and has a dual $\mathfrak{A}$-module structure (in addition to the one described above) defined by

$$(9)\qquad \begin{cases} (A\xi)(A_1, \ldots, A_k) = \xi(A_1, \ldots, A_{k-1}, A_k A), \\ (\xi A)(A_1, \ldots, A_k) = \xi(A_1, \ldots, A_k) A. \end{cases}$$

This process can be applied, with $\mathcal{M}$ replaced by the module $C_c^p(\mathfrak{A}, \mathcal{M})$ defined in (8), to give $C_c^k(\mathfrak{A}, C_c^p(\mathfrak{A}, \mathcal{M}))$ the structure of a dual $\mathfrak{A}$-module. The equation

$$\bar{\xi}(A_1, \ldots, A_{k+p}) = \xi(A_1, \ldots, A_k)(A_{k+1}, \ldots, A_{k+p})$$

defines an isometric linear isomorphism $\xi \to \bar{\xi}$ from $C_c^k(\mathfrak{A}, C_c^p(\mathfrak{A}, \mathcal{M}))$ onto $C_c^{k+p}(\mathfrak{A}, \mathcal{M})$. This isomorphism is weak * bicontinuous, since it is the adjoint of the natural isomorphism between the appropriate predual spaces, arising from the associativity of tensor products. It can therefore be used to transfer the dual $\mathfrak{A}$-module structure from $C_c^k(\mathfrak{A}, C_c^p(\mathfrak{A}, \mathcal{M}))$ to $C_c^{k+p}(\mathfrak{A}, \mathcal{M})$. When this is done, the module operations are given by

$$(10)\qquad \begin{cases} (A\xi)(A_1, \ldots, A_{k+p}) = \xi(A_1, \ldots, A_{k-1}, A_k A, A_{k+1}, \ldots, A_{k+p}), \\ (\xi A)(A_1, \ldots, A_{k+p}) \\ \quad = \xi(A_1, \ldots, A_k, AA_{k+1}, A_{k+2}, \ldots, A_{k+p}) \\ \quad + \sum_{j=1}^{p-1} (-1)^j \xi(A_1, \ldots, A_k, A, A_{k+1}, \ldots, A_{k+j} A_{k+j+1}, \ldots, A_{k+p}) \\ \quad + (-1)^p \xi(A_1, \ldots, A_k, A, A_{k+1}, \ldots, A_{k+p-1}) A_{k+p} \end{cases}$$

for ξ in $C_c^{k+p}(\mathfrak{A}, \mathcal{M})$ and $A, A_1, \ldots, A_{k+p}$ in $\mathfrak{A}$.

If $\mathcal{M}$ is a two-sided dual module for a Banach algebra $\mathfrak{A}$, and $\mathcal{N}$ is a weak * closed $\mathfrak{A}$-submodule of $\mathcal{M}$, then $\mathcal{N}$ is itself a dual $\mathfrak{A}$-module; for $\mathcal{N}$ is (isometrically isomorphic to) the dual space of a quotient space of $\mathcal{M}_*$, and the weak * topology on $\mathcal{N}$ coincides with its relative weak * topology as a subspace of $\mathcal{M}$.

THEOREM 4.1. — *If $\mathcal{B}$ is a closed amenable subalgebra of a Banach algebra $\mathfrak{A}$, $\mathcal{M}$ is a two-sided dual $\mathfrak{A}$-module and $\rho \in Z_c^n(\mathfrak{A}, \mathcal{M})$, there is a ξ in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ such that*

$$(\rho - \Delta\xi)(A_1, \ldots, A_n) = 0 \quad \textit{if any one of } A_1, \ldots, A_n \textit{ lies in } \mathcal{B}.$$

Proof. — We construct, inductively, $\xi_1, \ldots, \xi_n$ in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ such that $(\rho - \Delta\,\xi_k)(A_1, \ldots, A_n) = 0$ if any one of $A_1, \ldots, A_k$ lies in $\mathcal{B}$. The conclusion of the theorem then follows, with $\xi = \xi_n$.

To constructs ξ_1, we consider $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ as a dual $\mathfrak{A}$-module (hence, also, a dual $\mathcal{B}$-module), with the structure defined by (8) when $p = n - 1$. We define δ in $C_c^1(\mathcal{B}, C_c^{n-1}(\mathfrak{A}, \mathcal{M}))$ by

$$\delta(B)(A_2, \ldots, A_n) = \rho(B, A_2, \ldots, A_n) \qquad (B \in \mathcal{B};\, A_2, \ldots, A_n \in \mathfrak{A}).$$

By use of the coboundary formula and (8), we obtain

$$\begin{aligned} 0 &= (\Delta\rho)(B_0, B_1, A_2, \ldots, A_n) \\ &= (B_0\,\delta(B_1) - \delta(B_0\,B_1) + \delta(B_0)\,B_1)(A_2, \ldots, A_n), \end{aligned}$$

for all B_0, B_1 in $\mathcal{B}$ and $A_2, \ldots, A_n$ in $\mathfrak{A}$. Thus $\delta \in Z_c^1(\mathcal{B}, C_c^{n-1}(\mathfrak{A}, \mathcal{M}))$.

Since $\mathcal{B}$ is amenable, there exists ξ_1 in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ such that $\delta(B) = B\xi_1 - \xi_1 B$ $(B \in \mathcal{B})$. With B in $\mathcal{B}$ and $A_2, \ldots, A_n$ in $\mathfrak{A}$, we deduce from (8) that

$$\begin{aligned} \rho(B, A_2, \ldots, A_n) &= \delta(B)(A_2, \ldots, A_n) \\ &= (B\,\xi_1)(A_2, \ldots, A_n) - (\xi_1\,B)(A_2, \ldots, A_n) \\ &= (\Delta\xi_1)(B, A_2, \ldots, A_n). \end{aligned}$$

This proves the existence of a suitable cochain ξ_1.

Suppose now that $1 \leq k < n$, and a suitable cochain ξ_k has been constructed. With $\rho - \Delta\xi_k$ denoted by σ,

$$(11) \qquad \sigma(A_1, \ldots, A_n) = 0 \quad \text{if any one of } A_1, \ldots, A_k \text{ lies in } \mathcal{B}.$$

In order to continue the inductive process (and so complete the proof of the theorem), it suffices to contruct ζ in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ such that $\sigma - \Delta\zeta$ $[= \rho - \Delta(\xi_k + \zeta) = \rho - \Delta\xi_{k+1}$, with $\xi_{k+1} = \xi_k + \zeta]$ vanishes whenever any one of its first $k + 1$ arguments lies in $\mathcal{B}$. To this end, we consider $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ as a dual $\mathfrak{A}$-module (hence, also, a dual $\mathcal{B}$-module), with the structure defined by (10) when $p = n - 1 - k$. In the case $k = n - 1$, we have $p = 0$, and (10) is interpreted as (9).

Let $\mathcal{N}$ denote the linear subspace of $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ consisting of all cochains η satisfying the conditions

(12) $\eta(A_1, \ldots, A_{n-1}) = 0$ if any one of $A_1, \ldots, A_k$ lies in $\mathcal{B}$,

and

(13) $$B[\eta(A_1, \ldots, A_{n-1})] = \eta(BA_1, A_2, \ldots, A_{n-1}),$$

(14) $$\eta(A_1, \ldots, A_{j-1}, A_j B, A_{j+1}, \ldots, A_{n-1}) = \eta(A_1, \ldots, A_j, BA_{j+1}, A_{j+2}, \ldots, A_{n-1})$$

whenever $A_1, \ldots, A_{n-1} \in \mathfrak{A}$, $1 \leqq j < k$, and $B \in \mathcal{B}$. A routine argument shows that $\mathcal{N}$ is weak * closed in $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$, and is a $\mathcal{B}$-submodule of $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$ [recall that the module structure is defined by (10)]. Thus $\mathcal{N}$ is a dual $\mathcal{B}$-module.

If $\eta \in \mathcal{N}$, $A_1, \ldots, A_n \in \mathfrak{A}$, $1 \leqq j \leqq k$ and $A_j \in \mathcal{B}$, it results from (12), (13) and (14) that all terms but the j-th and $(j+1)$-st in the formal expansion of $(\Delta\eta)(A_1, \ldots, A_n)$ are zero, while the two remaining terms have sum zero. Thus

(15) $(\Delta\eta)(A_1, \ldots, A_n) = 0$ if $\eta \in \mathcal{N}$ and any one of $A_1, \ldots, A_k$ is in $\mathcal{B}$.

We define δ in $C_c^1(\mathcal{B}, C_c^{n-1}(\mathfrak{A}, \mathcal{M}))$ by

$$\delta(B)(A_1, \ldots, A_{n-1}) = \sigma(A_1, \ldots, A_k, B, A_{k+1}, \ldots, A_{n-1}).$$

From (11) and the relation $(\Delta\sigma)(A_1, \ldots, A_{k+1}, B, A_{k+2}, \ldots, A_n) = 0$ (with $A_1, \ldots, A_n$ in $\mathfrak{A}$, B in $\mathcal{B}$, and one of $A_1, \ldots, A_k$ in $\mathcal{B}$), it follows that $\delta(B)$ satisfies (12), (13) and (14). Thus $\delta(B) \in \mathcal{N}$, and $\delta \in C_c^1(\mathcal{B}, \mathcal{N})$. By use of the coboundary formula, (11) and (10), we obtain

$$\begin{aligned} 0 &= (\Delta\sigma)(A_1, \ldots, A_k, B_1, B_2, A_{k+1}, \ldots, A_{n-1}) \\ &= (-1)^k (B_1 \delta(B_2) - \delta(B_1 B_2) + \delta(B_1) B_2)(A_1, \ldots, A_{n-1}), \end{aligned}$$

for all B_1, B_2 in $\mathcal{B}$ and $A_1, \ldots, A_{n-1}$ in $\mathfrak{A}$. Thus $\delta \in Z_c^1(\mathcal{B}, \mathcal{N})$. Since $\mathcal{B}$ is amenable, there exists η in $\mathcal{N}[\subseteq C_c^{n-1}(\mathfrak{A}, \mathcal{M})]$ such that $\delta(B) = B\eta - \eta B$ $(B \in \mathcal{B})$. With $A_1, \ldots, A_{n-1}$ in $\mathfrak{A}$ and B in $\mathcal{B}$, it follows from (12) and (10), that

$$\begin{aligned} &(\Delta\eta)(A_1, \ldots, A_k, B, A_{k+1}, \ldots, A_{n-1}) \\ &\quad = (-1)^k [\eta(A_1, \ldots, A_{k-1}, A_k B, A_{k+1}, \ldots, A_{n-1}) \\ &\qquad - \eta(A_1, \ldots, A_k, BA_{k+1}, A_{k+2}, \ldots, A_{n-1}) + \cdots \\ &\qquad \pm \eta(A_1, \ldots, A_k, B, A_{k+1}, \ldots, A_{n-3}, A_{n-2} A_{n-1}) \\ &\qquad \mp \eta(A_1, \ldots, A_k, B, A_{k+1}, \ldots, A_{n-2}) A_{n-1}] \\ &\quad = (-1)^k (B\eta - \eta B)(A_1, \ldots, A_{n-1}) \\ &\quad = (-1)^k \delta(B)(A_1, \ldots, A_{n-1}) \\ &\quad = (-1)^k \sigma(A_1, \ldots, A_k, B, A_{k+1}, \ldots, A_{n-1}). \end{aligned}$$

The last equation, together with (11) and (15), shows that, if $\zeta = (-1)^k \eta$, then $\sigma - \Delta\zeta$ vanishes when any of its first $k+1$ arguments lies in $\mathcal{B}$. As noted above, this completes the proof of the theorem.

5. Normal cohomology

Our main purpose in this section is to prove that, in a sense explained below, $H^n_c(\mathfrak{A}, \mathcal{M}) = H^n_w(\mathfrak{A}, \mathcal{M})$ whenever $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}^-$-module. We begin by stating two auxiliary results which slightly generalise Lemmas 3.1 and 3.2 in [7], and are proved by the same methods.

LEMMA 5.1. — *If $\mathfrak{A}$ is a Banach algebra with centre $\mathcal{C}$, $\mathcal{D}$ is a subalgebra of $\mathcal{C}$, $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module, $1 \leq k \leq n$, and ρ in $Z^n_c(\mathfrak{A}, \mathcal{M})$ vanishes whenever any of its first k arguments lies in $\mathcal{D}$, then*

$$\rho(A_1, \ldots, A_{j-1}, DA_j, A_{j+1}, \ldots, A_n) = D\rho(A_1, \ldots, A_n)$$

whenever $1 \leq j \leq k$, $D \in \mathcal{D}$ and $A_1, \ldots, A_n \in \mathfrak{A}$.

LEMMA 5.2. — *If $\mathfrak{A}$ is a Banach algebra with centre $\mathcal{C}$, $\mathcal{D}$ is a subalgebra of $\mathcal{C}$, $\mathcal{M}$ is a two-sided Banach $\mathfrak{A}$-module, $n \geq 1$, and ρ in $Z^n_c(\mathfrak{A}, \mathcal{M})$ vanishes whenever any of its arguments lies in $\mathcal{D}$, then*

$$\rho(A_1, \ldots, A_{j-1}, DA_j, A_{j+1}, \ldots, A_n) = D\rho(A_1, \ldots, A_n) = \rho(A_1, \ldots, A_n)D$$

whenever $1 \leq j \leq n$, $D \in \mathcal{D}$ and $A_1, \ldots, A_n \in \mathfrak{A}$.

LEMMA 5.3. — *Suppose that $\mathfrak{A}$ is a unital C^*-algebra acting on a Hilbert space $\mathcal{H}$, $\mathcal{V}$ is a finite subgroup of the unitary group of the centre $\mathcal{C}$ of $\mathfrak{A}$, $\mathcal{D}$ is the subalgebra of $\mathcal{C}$, generated (linearly) by $\mathcal{V}$, and $\mathcal{B}$ is a closed amenable subalgebra of $\mathfrak{A}$. If $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}$-module, $n \geq 1$, and ρ in $Z^n_w(\mathfrak{A}, \mathcal{M})$ vanishes whenever any of its arguments lies in $\mathcal{B}$, there is a ξ in $C^{n-1}_w(\mathfrak{A}, \mathcal{M})$ such that $\rho - \Delta\xi$ vanishes whenever any of its arguments lies in either $\mathcal{D}$ or $\mathcal{B}$.*

Proof. — Since $\mathcal{V}$ is a finite group, it has a unique invariant mean μ. With φ a mapping from $\mathcal{V}$ into $\mathcal{M}$ and $\bar{\mu} : l_\infty(\mathcal{V}, \mathcal{M}) \to \mathcal{M}$ defined as in [7] (Lemma 3.3), we have

$$\bar{\mu}(\varphi) = q^{-1} \sum_{V \in \mathcal{V}} \varphi(V),$$

where q is the order of $\mathcal{V}$. We refer to $\bar{\mu}(\varphi)$ as the mean of φ.

The argument that follows is closely analogous to the proof of [7] (Theorem 3.4). With ξ_0 the zero element of $C^{n-1}_w(\mathfrak{A}, \mathcal{M})$, we define

$\xi_1, \ldots, \xi_n$ in $C_w^{n-1}(\mathfrak{A}, \mathfrak{M})$, inductively, as follows. Having constructed ξ_k, let $\sigma = \rho - \Delta\xi_k$ $[\in Z_w^n(\mathfrak{A}, \mathfrak{M})]$, and set

$$(16) \qquad \eta(A_1, \ldots, A_{n-1}) = q^{-1} \sum_{V \in \mathcal{V}} V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}),$$

so that $\eta \in C_w^{n-1}(\mathfrak{A}, \mathfrak{M})$ and $\eta(A_1, \ldots, A_{n-1})$ is the mean of the mapping

$$V \to V^* \sigma(A_1, \ldots, A_k, V, A_{k+1}, \ldots, A_{n-1}) : \quad \mathcal{V} \to \mathfrak{M}.$$

We then define ξ_{k+1} to be $\xi_k + (-1)^k \eta$ $[\in C_w^{n-1}(\mathfrak{A}, \mathfrak{M})]$. Exactly as in the proof of [7] (Theorem 3.4), we can show, by induction on k and making use of Lemmas 5.1 and 5.2, that

$$(17) \quad (\rho - \Delta\xi_k)(A_1, \ldots, A_n) = 0 \quad \text{if any one of } A_1, \ldots, A_k \text{ lies in } \mathcal{D}.$$

We claim also that

$$(18) \quad (\rho - \Delta\xi_j)(A_1, \ldots, A_n) = 0 \quad \text{if any one of } A_1, \ldots, A_n \text{ lies in } \mathcal{B}.$$

Once (18) is proved, the conclusion of the theorem follows, with $\xi = \xi_n$.

Since ρ vanishes when any of its arguments lies in $\mathcal{B}$, (18) is equivalent to

$$(19) \quad (\Delta\xi_j)(A_1, \ldots, A_n) = 0 \quad \text{if any one of } A_1, \ldots, A_n \text{ lies in } \mathcal{B}.$$

This last condition is obviously satified when $j = 0$, since ξ_0 is the zero cochain. With $0 \leq k < n$, we make the inductive assumption that (19) holds when $j = k$. In order to show that (19) is true also when $j = k + 1$, it now suffices to prove that

$$(20) \quad (\Delta\eta)(A_1, \ldots, A_n) = 0 \quad \text{if any one of } A_1, \ldots, A_n \text{ lies in } \mathcal{B},$$

since $\xi_{k+1} = \xi_k + (-1)^k \eta$.

Since $\sigma = \rho - \Delta\xi_k$,

$$(21) \quad \sigma(A_1, \ldots, A_n) = 0 \text{ when any one of } A_1, \ldots, A_n \text{ lies in } \mathcal{B};$$

for ρ and $\Delta\xi_k$ both have this property (the latter, by our inductive assumption). By considering $(\Delta\sigma)(A_0, \ldots, A_n)$ when some A_j is in $\mathcal{B}$, we deduce from (21) that

$$(22) \quad A_0 \sigma(A_1, \ldots, A_n) = \sigma(A_0 A_1, A_2, \ldots, A_n) \qquad \text{if} \quad A_0 \in \mathcal{B},$$

$$(23) \quad \begin{cases} \sigma(A_0, \ldots, A_{j-2}, A_{j-1} A_j, A_{j+1}, \ldots, A_n) \\ \quad = \sigma(A_0, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_n) \\ \qquad \text{if} \quad 0 < j < n \quad \text{and} \quad A_j \in \mathcal{B}, \end{cases}$$

$$(24) \quad \sigma(A_0, \ldots, A_{n-2}, A_{n-1} A_n) = \sigma(A_0, \ldots, A_{n-1}) A_n \qquad \text{if} \quad A_n \in \mathcal{B}.$$

From equations (21), ..., (24) and the definition (16) of η, it follows that

(25) $\eta(A_1, \ldots, A_{n-1}) = 0$ if any one of $A_1, \ldots, A_{n-1}$ lies in $\mathcal{B}$,

(26) $A_0\,\eta(A_1, \ldots, A_{n-1}) = \eta(A_0 A_1, A_2, \ldots, A_{n-1})$ if $A_0 \in \mathcal{B}$,

(27) $$\left\{\begin{array}{l}\eta(A_0, \ldots, A_{j-2}, A_{j-1} A_j, A_{j+1}, \ldots, A_{n-1}) \\ \quad = \eta(A_0, \ldots, A_{j-1}, A_j A_{j+1}, A_{j+2}, \ldots, A_{n-1}) \\ \qquad \text{if } 0 < j < n-1 \text{ and } A_j \in \mathcal{B},\end{array}\right.$$

(28) $\eta(A_0, \ldots, A_{n-3}, A_{n-2} A_{n-1}) = \eta(A_0, \ldots, A_{n-2})\, A_{n-1}$ if $A_{n-1} \in \mathcal{B}$.

Each of equations (25), and (26), (27), (28) in cases where $j \neq k$ in the condition $A_j \in \mathcal{B}$, follows from a single application of one of (21), ..., (24). When $j = k$, (26), (27) and (28) each requires two applications of appropriate equations from (22), (23), (24), and use of the fact that $A_k V = V A_k$ for each V in $\mathcal{V}$.

With $A_1, \ldots, A_n$ in $\mathfrak{A}$ and A_j in $\mathcal{B}$ for some j, it follows from (25), ..., (28) that all terms except the j-th and $(j+1)$-st in the expansion of $(\Delta\eta)(A_1, \ldots, A_n)$ are zero, while the two remaining terms have sum zero. This proves (20), and completes the proof of the lemma.

LEMMA 5.4. — *If φ is a faithful representation of a unital C*-algebra $\mathfrak{A}$, $\mathcal{B}$ is a closed amenable subalgebra of $\mathfrak{A}$, $\mathcal{M}$ is a two-sided dual normal $\varphi(\mathfrak{A})^-$-module, $n \geq 1$ and $\rho \in Z_c^n(\varphi(\mathfrak{A}), \mathcal{M})$, there exists ξ in $C_c^{n-1}(\varphi(\mathfrak{A}), \mathcal{M})$ such that $\rho - \Delta\xi \in Z_w^n(\varphi(\mathfrak{A}), \mathcal{M})$ and $\rho - \Delta\xi$ vanishes if any of its arguments lies in $\varphi(\mathcal{B})$.*

Proof. — Just as in the proof of [8] (Theorem 2.1), we way suppose that $\varphi(A) = AP$, where $\mathfrak{A}$ acting on $\mathcal{H}$ is the universal representation, and P is a projection in the centre $\mathcal{C}$ of $\mathfrak{A}^-$. Thus $\varphi(\mathfrak{A}) = \mathfrak{A} P$ and $\varphi(\mathfrak{A})^- = \mathfrak{A}^- P$. Let $\mathcal{V}$ be the subgroup $\{I, 2P - I\}$ of the unitary group of $\mathcal{C}$; so that the linear span of $\mathcal{V}$ is a subalgebra $\mathcal{D}$ of $\mathcal{C}$, containing P. Note that $\mathcal{M}$ becomes a two-sided dual normal $\mathfrak{A}^-$-module, such that $Pm = mP = m$ $(m \in \mathcal{M})$ if the left and right actions of $\mathfrak{A}^-$ on $\mathcal{M}$ are defined by

(29) $$A\,m = AP\,m, \qquad m\,A = m\,AP \qquad (A \in \mathfrak{A}^-, m \in \mathcal{M}).$$

Since φ is a faithful representation, $\mathcal{B} P\,[= \varphi(\mathcal{B})]$ is a closed amenable subalgebra of $\mathfrak{A} P$. In view of Theorem 4.1, it is sufficient to consider the case in which $\rho \in Z_c^n(\mathfrak{A} P, \mathcal{M})$, and ρ vanishes whenever any of its arguments lies in $\mathcal{B} P$. It follows that ρ_1, defined by

$$\rho_1(A_1, \ldots, A_n) = \rho(A_1 P, \ldots, A_n P) \qquad (A_1, \ldots, A_n \in \mathfrak{A})$$

is in $Z_c^n(\mathfrak{A}, \mathcal{M})$, and ρ_1 vanishes whenever any of its arguments lies in $\mathcal{B}$. Since $\mathfrak{A}$ acting on $\mathcal{H}$ is the universal representation, every norm continuous mapping from $\mathfrak{A}$ into $\mathcal{M}$ is ultraweak-weak * continuous; in particular, ρ_1 is (separately) continuous in this sense in each of its arguments [so $\rho_1 \in Z_w^n(\mathfrak{A}, \mathcal{M})$]. By Theorem 2.3, ρ_1 extends to an element $\bar{\rho}_1$ of $C_w^n(\mathfrak{A}^-, \mathcal{M})$. Since

$$(\Delta\bar{\rho}_1)(A_1, \dots, A_{n+1}) = 0 \tag{30}$$

whenever $A_1, \dots, A_{n+1} \in \mathfrak{A}$, it is readily verified (extending in one variable at a time, as in the proof of [8] (Theorem 2.1)) that (30) holds also when $A_1, \dots, A_{n+1}$ lies in $\mathfrak{A}^-$. Thus $\bar{\rho}_1 \in Z_w^n(\mathfrak{A}^-, \mathcal{M})$ and (by a similar but simpler continuity argument, extending in one variable at a time) $\bar{\rho}_1(A_1, \dots, A_n) = 0$ whenever $A_1, \dots, A_n \in \mathfrak{A}^-$ and some A_j lies in $\mathcal{B}$.

By Lemma 5.3 (with $\mathfrak{A}^-$ and $\bar{\rho}_1$ in place of $\mathfrak{A}$ and ρ respectively), there exists ξ_1 in $C_w^{n-1}(\mathfrak{A}^-, \mathcal{M})$ such that $\bar{\rho}_1 - \Delta\xi_1 [\in Z_w^n(\mathfrak{A}^-, \mathcal{M})]$ vanishes whenever any of its arguments lies in either $\mathcal{D}$ or $\mathcal{B}$. Since $P \in \mathcal{D}$, it follows, from Lemma 5.2, that

$$\begin{aligned}(\bar{\rho}_1 - \Delta\xi_1)(A_1, \dots, A_n) &= P(\bar{\rho}_1 - \Delta\xi_1)(A_1, \dots, A_n) \\ &= (\bar{\rho}_1 - \Delta\xi_1)(PA_1, \dots, PA_n),\end{aligned} \tag{31}$$

whenever $A_1, \dots, A_n \in \mathfrak{A}^-$. The faithful representation $A \to AP$ of $\mathfrak{A}$ is isometric, so we can define ξ in $C_c^{n-1}(\mathfrak{A}P, \mathcal{M})$ by

$$\xi(A_1P, \dots, A_{n-1}P) = \xi_1(A_1, \dots, A_{n-1}) \qquad (A_1, \dots, A_{n-1} \in \mathfrak{A}).$$

Thus, by (29),

$$\begin{aligned}&(\rho - \Delta\xi)(A_1P, \dots, A_nP) \\ &= \rho_1(A_1, \dots, A_n) - A_1P\xi(A_2P, \dots, A_nP) \\ &\quad + \xi(A_1A_2P, A_3P, \dots, A_nP) - \dots \\ &\quad \pm \xi(A_1P, \dots, A_{n-2}P, A_{n-1}A_nP) \mp \xi(A_1P, \dots, A_{n-1}P)A_nP \\ &= \bar{\rho}_1(A_1, \dots, A_n) - A_1\xi_1(A_2, \dots, A_n) \\ &\quad + \xi_1(A_1A_2, A_3, \dots, A_n) - \dots \\ &\quad \pm \xi_1(A_1, \dots, A_{n-2}, A_{n-1}A_n) \mp \xi_1(A_1, \dots, A_{n-1})A_n \\ &= (\bar{\rho}_1 - \Delta\xi_1)(A_1, \dots, A_n).\end{aligned}$$

Since this last quantity is zero when any A_j lies in $\mathcal{B}$, $\rho - \Delta\xi$ vanishes when any of its arguments lies in $\mathcal{B}P [= \varphi(\mathcal{B})]$. Furthermore, it now follows from (31) that $\rho - \Delta\xi = (\bar{\rho}_1 - \Delta\xi_1) \mid \mathfrak{A}P$, the restriction to $\mathfrak{A}P$ of $\bar{\rho}_1 - \Delta\xi_1 [\in Z_w^n(\mathfrak{A}^-, \mathcal{M})]$; and therefore $\rho - \Delta\xi \in Z_w^n(\mathfrak{A}P, \mathcal{M})$.

LEMMA 5.5. — *If φ is a faithful representation of a unital C*-algebra $\mathfrak{A}$ and $\mathcal{M}$ is a two-sided dual normal $\varphi(\mathfrak{A})^-$-module, then*

$$B_c^n(\varphi(\mathfrak{A}), \mathcal{M}) \cap Z_w^n(\varphi(\mathfrak{A}), \mathcal{M}) = B_w^n(\varphi(\mathfrak{A}), \mathcal{M}) \qquad (n = 1, 2, \ldots).$$

Proof. — Just as in the proof of the preceding lemma, we may assume that $\mathfrak{A}$ acting on $\mathcal{H}$ is the universal representation and that $\varphi(A) = AP$ $(A \in \mathfrak{A})$, for some central projection P in $\mathfrak{A}^-$. Furthermore, $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}^-$-module, with the action of $\mathfrak{A}^-$ on $\mathcal{M}$ defined by (29).

Suppose that $\rho \in Z_w^n(\mathfrak{A}P, \mathcal{M})$ and $\rho = \Delta\xi$ for some ξ in $C_c^{n-1}(\mathfrak{A}P, \mathcal{M})$. We construct ρ_1 in $Z_w^n(\mathfrak{A}, \mathcal{M})$ and its extension $\bar{\rho}_1$ in $Z_w^n(\mathfrak{A}^-, \mathcal{M})$, exactly as in the proof of Lemma 5.4. Furthermore, the equation

$$\xi_1(A_1, \ldots, A_{n-1}) = \xi(A_1P, \ldots, A_{n-1}P) \qquad (A_1, \ldots, A_{n-1} \in \mathfrak{A})$$

defines an element ξ_1 of $C_c^{n-1}(\mathfrak{A}, \mathcal{M})$. Since $\mathfrak{A}$ acting on $\mathcal{H}$ is the universal representation, ξ_1 is (separately) ultraweak-weak * continuous in each argument, so $\xi_1 \in C_w^{n-1}(\mathfrak{A}, \mathcal{M})$. By Theorem 2.3, ξ_1 extends to an element $\bar{\xi}_1$ of $C_w^{n-1}(\mathfrak{A}^-, \mathcal{M})$. It is readily verified that, for $A_1, \ldots, A_n$ in $\mathfrak{A}$

$$(\bar{\rho}_1 - \Delta\bar{\xi}_1)(A_1, \ldots, A_n) = (\rho - \Delta\xi)(A_1P, \ldots, A_nP) = 0.$$

Since $\bar{\rho}_1 - \Delta\bar{\xi} \in Z_w^n(\mathfrak{A}^-, \mathcal{M})$, it follows from continuity that $\bar{\rho}_1 = \Delta\bar{\xi}_1$. Hence $\rho_2 = \Delta\xi_2$, where ρ_2 in $Z_w^n(\mathfrak{A}P, \mathcal{M})$ and ξ_2 in $C_w^{n-1}(\mathfrak{A}P, \mathcal{M})$ are obtained by restricting $\bar{\rho}_1$ and $\bar{\xi}_1$ to $\mathfrak{A}P$ $(\subseteq \mathfrak{A}^-P \subseteq \mathfrak{A}^-)$.

We assert that $\rho_2 = \rho$. For this, note first that ρ extends to an element $\bar{\rho}$ of $C_w^n(\mathfrak{A}^-P, \mathcal{M})$, by Theorem 2.3. With σ defined by $\sigma(A_1, \ldots, A_n) = \bar{\rho}(A_1P, \ldots, A_nP)$ when $A_1, \ldots, A_n \in \mathfrak{A}^-$, σ lies in $C_w^n(\mathfrak{A}^-, \mathcal{M})$, as does $\bar{\rho}_1$. Since $\sigma \mid \mathfrak{A} = \rho_1 = \bar{\rho}_1 \mid \mathfrak{A}$, it follows by ultraweak-weak * continuity that $\sigma = \bar{\rho}_1$. With $A_1, \ldots, A_n$ in $\mathfrak{A}$,

$$\begin{aligned}\bar{\rho}_1(A_1P, \ldots, A_nP) &= \sigma(A_1P, \ldots, A_nP) \\ &= \bar{\rho}(A_1P, \ldots, A_nP) = \rho(A_1P, \ldots, A_nP).\end{aligned}$$

Thus $\rho = \bar{\rho}_1 \mid \mathfrak{A}P = \rho_2 = \Delta\xi_2$.

We have now shown that, if $\rho \in Z_w^n(\mathfrak{A}P, \mathcal{M})$ and $\rho = \Delta\xi$ for some ξ in $C_c^{n-1}(\mathfrak{A}P, \mathcal{M})$, then $\rho = \Delta\xi_2$ for some ξ_2 in $C_w^{n-1}(\mathfrak{A}P, \mathcal{M})$; in other words,

$$Z_w^n(\mathfrak{A}P, \mathcal{M}) \cap B_c^n(\mathfrak{A}P, \mathcal{M}) \subseteq B_w^n(\mathfrak{A}P, \mathcal{M}).$$

The reverse inclusion is apparent, so the theorem is proved.

Suppose $\mathfrak{A}$ is a unital C^*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is two-sided dual normal $\mathfrak{A}^-$-module. For each ρ in $Z^n_w(\mathfrak{A}, \mathcal{M})$, the coset $\rho + B^n_w(\mathfrak{A}, \mathcal{M})$ [$\in H^n_w(\mathfrak{A}, \mathcal{M})$] is a subset of the coset $\rho + B^n_c(\mathfrak{A}, \mathcal{M})$ [$\in H^n_c(\mathfrak{A}, \mathcal{M})$]. Hence there is a natural homomorphism

$$\Phi: \quad \rho + B^n_w(\mathfrak{A}, \mathcal{M}) \to \rho + B^n_c(\mathfrak{A}, \mathcal{M})$$

from $H^n_w(\mathfrak{A}, \mathcal{M})$ into $H^n_c(\mathfrak{A}, \mathcal{M})$. This homomorphism is one to one by Lemma 5.5, and has range the whole of $H^n_c(\mathfrak{A}, \mathcal{M})$ by Lemma 5.4. We have therefore proved the following result.

THEOREM 5.6. — *If $\mathfrak{A}$ is a unital C^*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}^-$-module, then*

$$H^n_w(\mathfrak{A}, \mathcal{M}) \simeq H^n_c(\mathfrak{A}, \mathcal{M}).$$

Let $\mathcal{R}$ be a von Neumann algebra. If it is the case that $H^2_w(\mathcal{R}, \mathcal{R}) = 0$, then we have the following Tauberian result : if $\xi \in C^1_c(\mathcal{R}, \mathcal{R})$ and $\Delta\xi \in Z^2_w(\mathcal{R}, \mathcal{R})$, then $\xi \in C^1_w(\mathcal{R}, \mathcal{R})$. To prove this, note that $\Delta\xi \in Z^2_w(\mathcal{R}, \mathcal{R}) = B^2_w(\mathcal{R}, \mathcal{R})$, and so $\Delta\xi = \Delta\xi_1$ for some ξ_1 in $C^1_w(\mathcal{R}, \mathcal{R})$. Since $\Delta(\xi - \xi_1) = 0$, $\xi - \xi_1$ is a derivation on $\mathcal{R}$, and so ultraweakly continuous by [6] (Lemma 3). Thus $\xi - \xi_1 \in C^1_w(\mathcal{R}, \mathcal{R})$, and

$$\xi = \xi_1 + (\xi - \xi_1) \in C^1_w(\mathcal{R}, \mathcal{R}).$$

In fact, the Tauberian result is true for any von Neumann algebra $\mathcal{R}$. We give two proofs, one based on the normal cohomology theory developed in this section, the other exploiting the characterization of normal linear mappings given in Corollary 3.4.

LEMMA 5.7. — *If $\mathcal{R}$ is a von Neumann algebra, $\xi \in C^1_c(\mathcal{R}, \mathcal{R})$ and $\Delta\xi \in Z^2_w(\mathcal{R}, \mathcal{R})$, then $\xi \in C^1_w(\mathcal{R}, \mathcal{R})$.*

First proof. — Since $\Delta\xi \in Z^2_w(\mathcal{R}, \mathcal{R}) \cap B^2_c(\mathcal{R}, \mathcal{R})$, it follows from Lemma 5.5 that $\Delta\xi = \Delta\xi_1$ for some ξ_1 in $C^1_w(\mathcal{R}, \mathcal{R})$. The argument, used in the paragraph preceding the statement of Lemma 5.7, now shows that $\xi \in C^1_w(\mathcal{R}, \mathcal{R})$.

Second proof. — By Corollary 3.4, it is sufficient to show that, if (E_α) is an orthogonal family of projections in $\mathcal{R}$, and $E = \sum E_\alpha$, then $\sum \xi(E_\alpha)$ converges ultraweakly to $\xi(E)$. By adding $I - E$ to the family (E_α), we may assume that $E = \sum E_\alpha = I$. All the finite subsums of $\sum \xi(E_\alpha)$ lie in the ball in $\mathcal{R}$ with centre 0 and radius $\| \xi \|$. The ultraweak topology on this ball is determined by the linear functionals $\omega_{x,y} : A \to \langle A x, y \rangle$

on $\mathcal{R}$, where y lies in the dense linear subspace generated algebraically by the ranges of the projections E_α. It is therefore sufficient to show that

$$\sum_\alpha \langle \xi (E_\alpha) x, y \rangle = \langle \xi (I) x, y \rangle$$

for such y; equivalently, that $\sum_\alpha E_\beta \xi (E_\alpha)$ converges ultraweakly to $E_\beta \xi (I)$, for each β. Now

$$E_\beta \xi (E_\alpha) = (\Delta\xi) (E_\beta, E_\alpha) + \xi (E_\beta E_\alpha) - \xi (E_\beta) E_\alpha.$$

Since $\Delta\xi$ is ultraweakly continuous in its second argument, we have

$$\sum_\alpha E_\beta \xi (E_\alpha) = (\Delta\xi) (E_\beta, I) + \xi (E_\beta) - \xi (E_\beta) = E_\beta \xi (I),$$

proving the lemma.

REMARK 5.8. — The second proof of Lemma 5.7 did not need the full force of the assumption that $\Delta\xi \in Z^2_w (\mathcal{R}, \mathcal{R})$, since no use was made of the ultraweak continuity of ξ in its first variable. By reasoning very similar to the second proof of Lemma 5.7, one can show that, if $\rho \in Z^2_c (\mathcal{R}, \mathcal{R})$ and ρ is ultraweakly continuous in one variable, then it is (separately) ultraweakly continuous in both variables [whence, $\rho \in Z^2_w (\mathcal{R}, \mathcal{R})$].

For higher cohomology, the obvious analogue of Lemma 5.7 is false. For example, if $\rho = \Delta\xi$ with ξ in $C^1_c (\mathcal{R}, \mathcal{R})$, but not in $C^1_w (\mathcal{R}, \mathcal{R})$, then $\rho \in C^2_c (\mathcal{R}, \mathcal{R})$ and $\Delta\rho$ $(= \Delta^2 \xi = 0)$ lies in $Z^3_w (\mathcal{R}, \mathcal{R})$; but, by Lemma 5.7, $\rho \notin C^2_w (\mathcal{R}, \mathcal{R})$.

6. Applications to norm continuous cohomology

THEOREM 6.1. — *If $\mathfrak{A}$ is a unital C*-algebra acting on a Hilbert space $\mathcal{H}$, and $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}^-$-module, then*

$$H^n_c (\mathfrak{A}, \mathcal{M}) \simeq H^n_c (\mathfrak{A}^-, \mathcal{M}) \ (n = 1, 2, \ldots).$$

Proof. — It follows, from Theorem 2.3, that, for $n = 1, 2, \ldots$, the restriction map $\iota_n : \eta \to \eta \mid \mathfrak{A}$ is a one to one linear mapping from $C^n_w (\mathfrak{A}^-, \mathcal{M})$ onto $C^n_w (\mathfrak{A}, \mathcal{M})$. Moreover, $\iota_n \Delta = \Delta \iota_{n-1}$ $(n = 1, 2, \ldots)$, provided ι_0 is interpreted as the identity mapping on $\mathcal{M}$ $[= C^0_c (\mathfrak{A}, \mathcal{M})]$. Thus ι_n maps $Z^n_w (\mathfrak{A}^-, \mathcal{M})$ onto $Z^n_w (\mathfrak{A}, \mathcal{M})$, $B^n_w (\mathfrak{A}^-, \mathcal{M})$ onto $B^n_w (\mathfrak{A}, \mathcal{M})$, and so induces an isomorphism between the quotient spaces $H^n_w (\mathfrak{A}^-, \mathcal{M})$ and $H^n_w (\mathfrak{A}, \mathcal{M})$. This, with Theorem 5.6, shows that $H^n_c (\mathfrak{A}^-, \mathcal{M}) \simeq H^n_c (\mathfrak{A}, \mathcal{M})$.

COROLLARY 6.2. — *If a von Neumann algebra $\mathcal{R}$ is the ultraweak closure of an amenable C^*-subalgebra $\mathfrak{A}$, then $H^n_c(\mathcal{R}, \mathcal{M}) = 0$ $(n = 1, 2, \ldots)$ for every two-sided dual normal $\mathcal{R}$-module $\mathcal{M}$.*

Proof. — Since $\mathfrak{A}$ is amenable, $H^n_c(\mathfrak{A}, \mathcal{M}) = 0$ $(n = 1, 2, \ldots)$; so the result follows from Theorem 6.1.

COROLLARY 6.3. — *If a von Neumann algebra $\mathcal{R}$ is the ultraweakly closed linear span of an amenable subgroup $\mathcal{V}$ of its unitary group, then $H^n_c(\mathcal{R}, \mathcal{M}) = 0$ $(n = 1, 2, \ldots)$ for every two-sided dual normal $\mathcal{R}$-module $\mathcal{M}$.*

Proof. — The norm closed linear span of $\mathcal{V}$ is an amenable C^*-algebra $\mathfrak{A}$ ([5], Proposition 7.8; [8], Theorem 3.3). Since $\mathfrak{A}^- = \mathcal{R}$, the result now follows from Corollary 6.2.

The following result generalises [8] (Theorem 3.1).

COROLLARY 6.4. — *If $\mathcal{R}$ is a hyperfinite von Neumann algebra, and $\mathcal{M}$ is a two-sided dual normal $\mathcal{R}$-module, then $H^n_c(\mathcal{R}, \mathcal{M}) = 0$ $(n=1, 2, \ldots)$.*

Proof. — Since $\mathcal{R}$ is the ultraweak closure of a uniformly hyperfinite C^*-subalgebra $\mathfrak{A}$, and (as noted in Section 4) such an algebra $\mathfrak{A}$ is amenable, the result follows from Corollary 6.2.

COROLLARY 6.5. — *If $\mathcal{R}$ is a type I von Neumann algebra, and $\mathcal{M}$ is a two-sided dual normal $\mathcal{R}$-module, then $H^n_c(\mathcal{R}, \mathcal{M}) = 0$ $(n = 1, 2, \ldots)$.*

Proof. — We can express $\mathcal{R}$ in the form $\Sigma \oplus \mathcal{R}_j \otimes \mathcal{C}_j$, where each $\mathcal{R}_j$ is a type I factor and each $\mathcal{C}_j$ is an abelian von Neumann algebra. By choosing a self-adjoint system of matrix units in $\mathcal{R}_j$, we associate with each element of $\mathcal{R}_j$ an infinite matrix. Given a finite subset F of the diagonal matrix units, we denote by $\mathcal{W}(F)$ the group of all unitary elements in $\mathcal{R}_j$ whose matrices have $+1$ in the diagonal position of each column corresponding to a diagonal matrix unit not in F, a single entry ± 1 somewhere in each other column, and zeros elsewhere. Since $\mathcal{W}(F)$ is finite and $\mathcal{W}(F_1 \cup F_2)$ contains $\mathcal{W}(F_1)$ and $\mathcal{W}(F_2)$, the union $\mathcal{W}_j$ of all $\mathcal{W}(F)$'s is an amenable group ([2], (F), p. 516). Moreover, the linear span of $\mathcal{W}_j$ contains each matrix unit and is therefore ultraweakly dense in $\mathcal{R}_j$. With $\mathcal{U}_j$ the (abelian) unitary group of $\mathcal{C}_j$, and

$$\mathcal{V}_j = \{ W \otimes U : W \in \mathcal{W}_j, U \in \mathcal{U}_j \},$$

$\mathcal{V}_j$ has linear span ultraweakly dense in $\mathcal{R}_j \otimes \mathcal{C}_j$. As a group, $\mathcal{V}_j$ is isomorphic to the direct product $\mathcal{W}_j \times \mathcal{U}_j$, and is therefore amenable ([2], (H), (E), p. 516). Finally, let $\mathcal{V}$ be the group of all unitary elements $\Sigma \oplus V_j$ in $\mathcal{R}$ for which each V_j lies in $\mathcal{V}_j$ and all but a finite set of V_j's are I. The linear span of $\mathcal{V}$ is ultraweakly dense in $\mathcal{R}$, and $\mathcal{V}$ is amenable since it is isomorphic to the restricted direct product of the $\mathcal{V}_j$'s ([2], (F″), p. 517). By Corollary 6.3, $H_c^n(\mathcal{R}, \mathcal{M}) = 0$.

For the case in which $\mathcal{M} = \mathcal{R}$, the result of Corollary 6.5 was first proved in [7] (Theorem 4.4) : another proof, by quite different methods, was given in [5] (Proposition 7.14). This latter argument can be applied, virtually unchanged, to give an alternative proof of Corollary 6.5 in its present generality.

We conclude by noting the following consequence of Theorem 6.1 and Corollary 6.5 : if $\mathfrak{A}$ is a unital C^*-algebra acting on a Hilbert space $\mathcal{H}$, $\mathfrak{A}^-$ is a type I von Neumann algebra and $\mathcal{M}$ is a two-sided dual normal $\mathfrak{A}^-$-module, then $H_c^n(\mathfrak{A}, \mathcal{M}) = 0$. In particular, $H_c^n(\mathfrak{A}, \mathfrak{A}^-) = 0$ and $H_c^n(\mathfrak{A}, \mathcal{B}(\mathcal{H})) = 0$ $(n = 1, 2, \ldots)$.

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